Morphism to projective space

Friday, November 22, 2024 10:47 AM

We are interested (a general version) of the following question.

<u>Question.</u> Let k be a field. X be a finite type scheme $/k$. When is X quasi-projective, i.e when is X isom to an <u>open subscheme</u> of a <u>closed subscheme</u> of $\mathbb{P}^n_k$.

Notation and Terminology:

$\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, For any scheme Y

$\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} Y$, $\mathcal{O}_{\mathbb{P}^n_Y}(1) = j^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$.

$$\begin{tikzcd} \mathbb{P}^n_Y \arrow[r, "j"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ Y \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

$\mathbb{P}^n_{\mathrm{Spec}\,A} =: \mathbb{P}^n_A$,

$\mathcal{L}$ invertible sheaf on X, $s \in \Gamma(X, \mathcal{L})$.

$D_s = \{ x \in X \mid s_x \notin m_x \mathcal{L}_x \}$. $D_s \subseteq X$ open.

<u>Note.</u> The map $\mathcal{O}_X \longrightarrow \mathcal{L}$, $1 \longmapsto s$ is an isom on D_s.

<u>Def.</u> For any $t \in \Gamma(V, \mathcal{L})$, $V \supseteq D_s$, $t/s \in \mathcal{O}_X(U)$ is the unique elt s.t $t|_U = t/s \cdot s|_U$

§ <u>Morphism to $\mathbb{P}^n$</u>

Let A be a ring.

<u>Thm.</u> Let X be an A-scheme (i.e fix a morphism $X \to \mathrm{Spec}\,A$) $\mathcal{L}$ be an invertible $\mathcal{O}_X$-mod. $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ s.t $X = \bigcup_{i=0}^n D_{s_i}$.

- There is a A-scheme morphism
$$\varphi : X \longrightarrow \mathbb{P}^n_A$$
and an isom $\theta : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \xrightarrow{\sim} \mathcal{L}$ which sends $\varphi^*(x_i)$ to s_i

- Any A-scheme morphism
$$\psi : X \longrightarrow \mathbb{P}^n_A$$
such that there is an isom $\psi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \longrightarrow \mathcal{L}$ sending $\psi^*(x_i)$ to s_i must coincide with φ.

<u>Rmk.</u> Given $f : X_1 \longrightarrow X_2$ scheme map, $\mathcal{F}_2 \in \mathrm{Mod}_{\mathcal{O}_{X_2}}$, $t \in \Gamma(X_2, \mathcal{F}_2)$. Recall $f^* \mathcal{F}_2 := f^{-1} \mathcal{F}_2 \otimes \mathcal{O}_{X_1}$

Rmk. Given $f: X_1 \to X_2$ scheme map, $\mathcal{F}_2 \in Mod_{\mathcal{O}_{X_2}}$, $t \in \Gamma(X_2, \mathcal{F}_2)$. Recall $f^*\mathcal{F}_2 := f^{-1}\mathcal{F}_2 \otimes \mathcal{O}_{X_1}$

t gives a section $f^{-1}(t) \in \Gamma(X_1, f^{-1}(\mathcal{F}_2))$

$f^*(t)$ denotes the section $f^{-1}(t) \otimes 1$.

More canonically, the adjunction between f^*, f_* gives a map (image of $id \in Hom(f^*\mathcal{F}_2, f^*\mathcal{F}_2)$), $\mathcal{F}_2 \to f_*(f^*\mathcal{F}_2)$.

f^*t is the image of t under this map.

Pf of Thm.

$$\mathbb{P}^n_A = \bigcup_{i=0}^n D_{x_i}, \quad D_{x_i} = D_+(x_i) = Spec\left(A\left[\frac{x_0}{x_i}, \dots \overset{\text{omitted}}{\frac{x_i}{x_i}}, \dots \frac{x_n}{x_i}\right]\right)$$

There is a unique A-scheme map φ_i for each i

$$X \supseteq D_{s_i} \longrightarrow D_{x_i}$$

induced by the A-alg map

$$A\left[\frac{x_0}{x_i}, \dots \widehat{x_i/x_i}, \dots x_n/x_i\right] \longrightarrow \Gamma(D_{s_i}, \mathcal{O}_X)$$

$$x_j/x_i \longmapsto s_j/s_i$$

$\varphi_i|_{D_{s_i}\cap D_{s_j}}, \varphi_j|_{D_{s_i}\cap D_{s_j}} : D_{s_i}\cap D_{s_j} \longrightarrow D_{x_i}\cap D_{x_j}$

both correspond to the same A-alg map

$$\begin{array}{c} A\left[\frac{x_0}{x_i}, x_1/x_i, \dots, x_i/x_i, \dots x_n/x_i\right]\left[(x_j/x_i)^{-1}\right] \\ \| \\ A\left[\frac{x_0}{x_j}, \dots, x_n/x_j\right]\left[\left(\frac{x_i}{x_j}\right)^{-1}\right] \end{array} \longrightarrow \Gamma(D_{s_i}\cap D_{s_j}, \mathcal{O}_X)$$

- Thus $\varphi_i|_{D_{s_i}\cap D_j} = \varphi_j|_{D_{s_i}\cap D_j}$
- Thus $\exists!\ \varphi: X \to \mathbb{P}^n_A$ such that $\varphi|_{D_{s_i}} = \varphi_i$.

The desired isom $\varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \to \mathcal{L}$ is constructed as follows. Note $\varphi^{-1}(D_{x_i}) = D_{s_i}$ by construction.

$$\begin{tikzcd} \theta_i : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1))|_{D_{s_i}} \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathcal{O}_{D_{s_i}}\cdot\varphi^*(x_i) \arrow[r] & \mathcal{O}_{D_{s_i}}\cdot s_i \arrow[r, "\sim"] & \mathcal{L}|_{D_{s_i}} \\ \varphi_i^*(\mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_{x_i}}) \arrow[r, "\sim"] & \varphi_i^*(\mathcal{O}_{\mathbb{P}^n_A}|_{D_{x_i}}\cdot x_i) \arrow[u, "\wr"] & & \end{tikzcd}$$

$$\varphi^*(x_i)|_{D_{s_i}} \longmapsto s_i|_{D_{s_i}}$$

Since $\theta_i|_{D_{s_i}\cap D_{s_j}} = \theta_j|_{D_{s_i}\cap D_j}$

$$\exists!\ \theta: \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \longrightarrow \mathcal{L} \text{ s.t. } \theta|_{D_{s_i}} = \theta_i$$

Clearly $\theta(\varphi^*(x_i)) = s_i\ \forall i$, $\left(\theta_j(\varphi^*(x_i)) = \theta_j\left(\frac{s_i}{s_j}\varphi^*(x_j)\right) = s_i/s_j \cdot s_j = s_i\right)$

For the uniqueness, given such a ψ,

The existence of the isom

$$\psi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \xrightarrow{\sim} \mathcal{L} \text{ sending } \psi^*(x_i) \text{ to } s_i$$

implies $\psi^{-1}(D_{x_i}) = D_{s_i}$. Indeed

The isom gives $\psi^*(x_i) \in m_x \psi^*(\mathcal{O}_{\mathbb{P}^n}(1))_x$

$$= \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)} \otimes m_x$$

The isom gives $\psi^*(x_i) \notin m_x \psi^*(\mathcal{O}_{\mathbb{P}^n}(1))_x$
$$= \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)} \otimes m_x$$
$\iff s_i \in m_x \mathcal{L}_x$.

But $\psi^*(x_i) = x_i \otimes 1 \in \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)} \otimes m_x$

$\iff x_i \in \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)}$ [ψ induces an injection $\frac{\mathcal{O}_{\mathbb{P}^n_A,\psi(x)}}{m_{\psi(x)}} \to \frac{\mathcal{O}_{X,x}}{m_x}$]

$\psi^{-1}(m_x) = m_{\psi(x)}$

Again the existence of the isom $\Rightarrow$

That map $A[x_0/x_i, \ldots, x_r/x_i] \to \Gamma(D_{x_i}, \mathcal{O}_X)$
That induces $\psi|_{D_{x_i}}$ sends x_j/x_i to s_j/s_i:

On D_{x_i} $\psi^*(x_j) = \psi^\#(x_j/x_i) \cdot \psi^*(x_i)$

So $\psi^\#(x_j/x_i) = \psi^*(x_j)/\psi^*(x_i)$
$$= s_j/s_i$$

Thus $\psi|_{D_{s_i}} = \varphi|_{D_{s_i}}$ $\forall i$. $\square$

End of 22.11.24 lecture

Rmk When X is f.t over an algebraically closed field.
The map induced by $s_0 \ldots s_n$ sends
a closed pt $x \in X$ to $[s_0(x) : s_1(x) : \quad : s_n(x)]$
$\in \mathbb{P}^n_k$.

Given an A-scheme X and $n \in \mathbb{N}$.

Thm There is a one to one correspondence

$$\frac{\left\{ \mathcal{L},\ \text{ordered tuple } (s_0, \ldots, s_n) \;\middle|\; \begin{array}{l}\mathcal{L} \text{ invertible} \\ s_i \in \Gamma(X, \mathcal{L}) \text{ s.t } \\ \bigcup_{i=0}^n D_{s_i} = X \end{array}\right\}}{\sim} \longrightarrow \left\{ \begin{array}{c} A\text{-scheme map} \\ X \to \mathbb{P}^n_A \end{array}\right\}$$

where $(\mathcal{L}_1, (s_0 \ldots s_n)) \sim (\mathcal{L}_2, (t_0 \ldots t_n))$
iff $\exists$ an isom of $\mathcal{O}_X$-mods
$\theta : \mathcal{L}_1 \to \mathcal{L}_2$ sending s_i to t_i $\forall i$.

Pf. Given $\varphi : X \to \mathbb{P}^n_A$ of A-schemes take
$$\mathcal{L}_\varphi = \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \quad \text{and} \quad s_i = \varphi^*(x_i).$$

The morphism induced by $(\mathcal{L}_\varphi, (s_0 \ldots, s_n))$ is indeed φ. The one-one correspondence is H.W.

Eg: $k = \mathbb{C}, \overline{\mathbb{Q}}, \ldots$ $\quad X = \mathbb{P}^1_k$, $\mathcal{L} = \mathcal{O}(2)$
$$\Gamma(\mathbb{P}^1, \mathcal{O}(2)) = k x^2 \oplus k xy \oplus k y^2.$$

get
$$\mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k$$
$$[x_0 : x_1] \longrightarrow [x_0^2 : x_1 x_2 : x_1^2]$$
The image is $V(Y^2 - XZ)$

§ Criterion for having an embedding

Recall. A *locally closed* subspace of a topological space is an intersection of a closed and open subset

Recall. A *locally closed* subspace of a topological space is an intersection of a closed and open subset with the induced topology.

Def. • A morphism of schemes $\varphi : X \to Y$ is called a locally closed immersion if (1) φ induces a homeo from X to a locally closed subset of Y and
(2) $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is surjective at all points of $y \in \varphi(X)$ — Locally closed immersion

• $\varphi : X \to Y$ is called a *closed* immersion if $\varphi(X) \subseteq Y$ is closed.

• $\varphi : X \to Y$ is called an open immersion if $\varphi(X) \subseteq Y$ is open and $\forall\, y \in \varphi(X)$, $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is an isom.

Prop. Every locally closed immersion $\varphi : X \to Y$ can be factored as
1) $X \xrightarrow{i} U \xrightarrow{j} Y$ where j is an open immersion and i is a closed immersion.

Pf. Write $\varphi(X) = Z \cap U$ where $Z \underset{\text{closed}}{\subseteq} Y$, $U \underset{\text{open}}{\subseteq} Y$.
Then φ factors as $X \xrightarrow{i} U \xrightarrow{j} Z$ as a scheme map, where U has the induced open scheme structure and j is the open immersion.
We claim that i is a closed embedding. Indeed i is a homeo onto the image and $i(X) = Z \cap U$ is closed in U.
Note that for a point in U which is not in $i(X)$, the stalk $(i_* \mathcal{O}_X)_y = \varphi_* (\mathcal{O}_X)_y = 0$. Indeed one can choose an open nbhd V of y such that $V \cap i(X) = \phi$. Then $i^{-1}(V) = \phi$. So $(i_* \mathcal{O}_X)_y = 0$.
We claim that the induced map
$$\mathcal{O}_U \to i_* \mathcal{O}_X \text{ is surjective.}$$
If $y \in i(X)$, $\mathcal{O}_{U,y} \to (i_* \mathcal{O}_X)_y$ is the same as the induced map $\mathcal{O}_{X,y} \to (\varphi_* \mathcal{O}_X)_y$ which is surjective by our assumption.
If $y \notin i(X)$, $\mathcal{O}_{U,y} \to (i_* \mathcal{O}_X)_y$ is surjective because the target is zero.

Rmk. In general it is not true that any locally closed immersion $\varphi : X \to Y$ can be factored as
$X \xrightarrow{j} Z \xrightarrow{i} Y$ where j is an open immersion and i is a closed immersion.
However we have the following.

Prop. Let $\varphi : X \to Y$ be a locally closed immersion such that $\varphi_* \mathcal{O}_X$ is quasi-coherent. Then $\varphi : X \to Y$ can be factored as
$$X \xrightarrow[j]{} Z \xrightarrow[i]{} Y$$
where j is an open immersion and i is closed.

Pf. Let $\mathcal{J} = \ker(\mathcal{O}_Y \to \varphi_* \mathcal{O}_X)$. For $y \in \varphi(X)$,

Pf. Let $\mathcal{J} = \ker(\mathcal{O}_Y \to \varphi_* \mathcal{O}_X)$. For $y \in \varphi(X)$,
since $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is surj, $\mathcal{J}_y \subseteq m_y$.
Since $\varphi_* \mathcal{O}_X$ is q.coh, $\mathcal{J}$ is q.coh.
So $V(\mathcal{J}) = \{ y \in Y \mid \mathcal{J}_y \subseteq m_y \}$ is closed in Y
$V(\mathcal{J}) \supseteq \varphi(X)$. Equip $V(\mathcal{J})$ with the scheme structure
given by $\mathcal{O}_Y/\mathcal{J}$, call the scheme Z

Note $\varphi(X)$ is open in $V(\mathcal{J})$. Since $\mathcal{J} \cdot \varphi_* \mathcal{O}_X = 0$,
the map $\varphi : X \to Y$ factors through the closed
immersion $\varphi : X \xrightarrow[j]{} Z \xrightarrow[i]{} Y$, i is the closed immersion.

We claim j is an open immersion. Indeed what
remains to check is that $\forall\, y \in \varphi(X) \subseteq Z$, the induced
map $\mathcal{O}_{Z,y} \to (j_* \mathcal{O}_X)_y$ is an isom. But
$\mathcal{O}_{Z,y} = \dfrac{\mathcal{O}_{Y,y}}{\mathcal{J}_y}$ is isom to $(j_* \mathcal{O}_X)_y$ for $y \in \varphi(X)$
via the map induced by φ and hence by j. ◻.

Def. Given a scheme Y. Sch_Y is the category
- whose objects are schemes X together with a morphism $X \to Y$
- Given $X_1 \to Y$, $X_2 \to Y$ in Sch_Y a Y-scheme morphism is a scheme map $\varphi : X_1 \to X_2$ s.t

$$\begin{tikzcd} X_1 \arrow[rr, "\varphi"] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$
commutes.

Def. • Let X be a Y scheme. An invertible $\mathcal{O}_X$
mod $\mathcal{L}$ is called Y-<u>very ample</u> if
$X \to Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd} \qquad *$$

where φ is locally closed and $\mathcal{L} \simeq \varphi^*(\mathcal{O}_{\mathbb{P}^n_Y}(1))$.

• Whenever the structure map $X \to Y$ has a factorization
as in $*$, we say <u>X is quasi projective over Y</u>.

Prop. A ring. Let X/A be proper. Any locally closed immersion
$\varphi : X \to \mathbb{P}^n_A$ is a closed immersion. Thus X is projective $/A$
$\Leftrightarrow$ X is quasi-projective $/A$.

Pf. $\mathbb{P}^n_A$ is separated, X/A proper. So
$\varphi(X)$ is closed in $\mathbb{P}^n_A$. ◻

• Whenever the structure map $X \to Y$ can be factored
as in $*$ with φ being a closed embedding,

- Whenever the structure map $X \to Y$ can be factored as in $*$ with φ being a closed embedding, the morphism $X \to Y$ is called projective.

<u>Thm.</u> Let X be a finite type scheme $/A$ = noetherian; $\mathcal{L}$ be an invertible sheaf on X.
$\mathcal{L}$ is ample $\Leftrightarrow$ $\mathcal{L}^m$ is very ample for some n.

<u>Pf.</u> In our setup a very ample invertible sheaf $\mathcal{L}'$ is ample.
Indeed, choose a locally closed embedding
$$X \underset{\varphi}{\hookrightarrow} \mathbb{P}^n_A \quad \text{s.t} \quad \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \simeq \mathcal{L}'.$$
Factor φ as $X \xrightarrow[j]{} Z \xrightarrow[i]{} \mathbb{P}^n_A$ where j is an open immersion and i is closed. Given $\mathcal{F} \in \mathrm{Coh}(X)$
$\exists$ $\mathcal{G} \in \mathrm{Coh}(Z)$ such that $j^*\mathcal{G} \simeq \mathcal{F}$. Take $\tilde{\mathcal{F}} = i_*\mathcal{G}$
Then $\varphi^*(\tilde{\mathcal{F}}) \simeq j^*(i^*(i_*\mathcal{G})) \simeq j^*\mathcal{G} \simeq \mathcal{F}$.

Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample, $\exists$ m_0 s.t $\forall$ $m \geq m_0$
$\tilde{\mathcal{F}} \otimes \mathcal{O}_{\mathbb{P}^n}(m)$ is globally gen. Then $\varphi^*(\tilde{\mathcal{F}} \otimes \mathcal{O}_{\mathbb{P}^n}(m))$
$\simeq \mathcal{F} \otimes \mathcal{L}'^m$
is also glob gen. So $\mathcal{L}'$ is ample.
$\mathcal{L}^m$ very ample $\Rightarrow$ $\mathcal{L}^m$ ample $\Rightarrow$ $\mathcal{L}$ ample.

Now assume $\mathcal{L}$ is ample.
<u>Step 1:</u> For each $x \in X$, there exists $s_x \in \Gamma(X, \mathcal{L}^{n_x})$ such that D_{s_x} is an affine open nbhd of x.

<u>Pf. of Step 1:</u> Fix $x \in X$ and an affine open nbhd U_x s.t $\mathcal{L}|_{U_x} \simeq \mathcal{O}_{U_x}$ of x. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf s.t
$V(\mathcal{I}) = X - U_x$. Choose n_x s.t $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated for $n \geq n_x$. So there exists $s_x \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{n_x}) \subseteq \Gamma(X, \mathcal{L}^{n_x})$
$s_x \notin m_x(\mathcal{I} \otimes \mathcal{L}^{n_x}) = m_x \mathcal{L}^{n_x}$ [$\because$ $\mathcal{I}_x = \mathcal{O}_{X,x}$]
Note $\forall$ $y \in X - U_x$, $(s_x)_y \in \mathcal{I}_y \otimes \mathcal{L}^{n_x}_y \subseteq m_y \mathcal{L}^{n_x}_y$

So $D_{s_x} \subseteq U_x$. Choose an isom $\mathcal{L}|_{U_x} \simeq \mathcal{O}_{U_x}$. Assume s_x goes to f_x
$D_{s_x} = D_{f_x}$ in U_x. So D_{s_x} is affine. ∎

Since X is quasi-compact, there is a finite covering
$$X = \bigcup_{i=1}^{r} D_{s_{x_i}}$$
Replace $\mathcal{L}$ by $\mathcal{L}^{n_{x_1} n_{x_2} \cdots n_{x_r}}$ and s_{x_i} by $s_{x_i}^{n_{x_1} \cdots n_{x_r}/n_{x_i}}$

Thus $\exists$ $s_{x_1} \ldots s_{x_r} \in \Gamma(X, \mathcal{L})$ such that
$$X = \bigcup_{i=1}^{r} D_{s_{x_i}} \quad \text{and each } D_{s_{x_i}} \text{ is affine}$$
For each $1 \leq i \leq r$, make a choice
$$\Gamma(D_{s_{x_i}}, X) = A[y_{ij} \mid 1 \leq j \leq n_i]$$
$\exists$ $l \in \mathbb{N}$ s.t $y_{ij} \otimes s_{x_i}^{l}$ extends to a global section t_{ij} of $\Gamma(X, \mathcal{L}^l)$ $\forall$ $1 \leq i \leq r$, $1 \leq j \leq n_i$

$\exists\ l \in \mathbb{N}$ s.t. $y_{ij} \otimes s_{x_i}^{l}$ extends to a global section t_{ij} of $\Gamma(X, \mathcal{L}^l)$, $\forall\ 1 \leq i \leq r$, $1 \leq j \leq n_i$

Consider the morphism φ to some $\mathbb{P}^N_A$ given by $\mathcal{L}, \{t_{ij}, s_{x_i}^{l}\}_{1\leq i\leq r,\ 1\leq j\leq n_i}$. $N = n_1+n_2+\dots+n_r+r-1$
$\mathbb{P}^N = \operatorname{Proj}(A[x_{ij}, y_i \mid 1\leq i\leq r,\ 1\leq j\leq n_i])$

Claim: φ is a locally closed immersion

Pf. Since $X = \bigcup_{i=1}^{r} D_{s_{x_i}^l}$, φ factors through $\bigcup_{i=1}^{r} D_{y_i} = V$

We show $\varphi : X \to V \subseteq \mathbb{P}^N$ is a closed immersion.

For that, enough to show
$D_{s_{x_i}} = \varphi^{-1}(D_{y_i}) \longrightarrow D_{y_i}$ is a closed immersion $\forall i$

$\iff$ $\Gamma(D_{y_i}, \mathbb{P}^N) \longrightarrow \Gamma(D_{s_{x_i}}, X)$ is surjective $\forall i$

$$\Gamma(D_{s_{x_i}}, X) = A[y_{ij} \mid 1 \leq j \leq r]$$

$$\frac{t_{ij}}{s_{x_i}^l} \longmapsto y_{ij} \quad [\because t_{ij} \text{ extends } y_{ij} \otimes s_{x_i}^l]. \qquad \square$$

End of 27.11.24 lecture

Prop: (i) A noetherian ring. $X \in Sch_A$ is quasi-projective iff X has an ample invertible sheaf.
(ii) If X is proper $/A$. Then X is projective $\iff$ X has an ample invertible sheaf.

§ Examples of projective morphisms. Blow-ups.

- X noetherian scheme, $\mathcal{S} = \bigoplus_{n\in\mathbb{N}} \mathcal{S}_n$ be an $\mathbb{N}$-graded quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that
- Each $\mathcal{S}_n$ is a q.coh $\mathcal{O}_X$-submod of $\mathcal{S}$
- The $\mathcal{O}_X$-alg structure on $\mathcal{S}$, comes from a ring map $\mathcal{O}_X \to \mathcal{S}_0$
- for each affine $\operatorname{Spec}(A) \subseteq X$
  $\mathcal{S}(\operatorname{Spec} A) = (S_0 \oplus S_1 \oplus S_2 \oplus \dots)^\sim$
  where $\oplus S_i$ is standard graded.

Prop. • There exists a scheme denoted $\operatorname{Proj}_X(\mathcal{S})$ with a map $\pi : \operatorname{Proj}_X(\mathcal{S}) \to X$ such that for each affine open $\operatorname{Spec}(A) \subseteq X$, (see page 166 Hart).
$\pi^{-1}(U) \xrightarrow{\sim} \operatorname{Proj}(S_0 \oplus S_1 \oplus \dots \oplus\ )$ where
$\mathcal{S}(\operatorname{Spec} A) = (\oplus S_i)^\sim$

- There exists an invertible sheaf denoted $\mathcal{O}(1)$ on $\operatorname{Proj}_X(\mathcal{S})$ s.t. $\mathcal{O}(1)|_{\operatorname{Spec}(A)} \xrightarrow{\sim} \widetilde{S(1)}$.
  where $S = \oplus S_i$.

- Given an invertible sheaf $\mathcal{L}$ and $\mathcal{S}$ as above define $\mathcal{S}' = \oplus \mathcal{S}_n \otimes \mathcal{L}^n$
  Then there is a natural isom
  $\varphi : \operatorname{Proj}(\mathcal{S}') \longrightarrow \operatorname{Proj}(\mathcal{S})$ such that

Then There is a natural isom
$$\varphi : \underline{\mathrm{Proj}}(\mathcal{S}') \longrightarrow \underline{\mathrm{Proj}}(\mathcal{S}) \text{ such That}$$
$$\mathcal{O}(1) \xrightarrow{\sim} \varphi^* \mathcal{O}(1) \otimes \pi^*(\mathcal{L}) \text{ where}$$
$\pi : \underline{\mathrm{Proj}}(\mathcal{S}) \to X$ is The natural projection map.

eg. $\mathcal{E} = \bigoplus^{r+1} \mathcal{O}_X \qquad r \in \mathbb{N}$

$\mathcal{S} = \mathrm{Sym}^\bullet(\mathcal{E}) = \mathcal{O}_X \oplus \mathrm{Sym}^1 \mathcal{O}_X^{\oplus r+1} \oplus \mathrm{Sym}^2 \cdots$

On $\mathrm{Spec}\, A$. $\mathcal{S}|_{\mathrm{Spec}\, A} \xrightarrow{\sim} A[T_0, \dots, T_r]$

"So" $\underline{\mathrm{Proj}}(\mathcal{S}) = X \times_A \mathbb{P}^n_A$.

Prop. If X has an ample invertible sheaf $\mathcal{L}$. Then $\pi : \underline{\mathrm{Proj}}(\mathcal{S}) \to X$ is projective.

Pf Choose $n \in \mathbb{N}$ s.t. $\mathcal{S}_1 \otimes \mathcal{L}^n$ is glob gen.

take $\mathcal{L}' = \mathcal{L}^n$.

Choose a surjection
$$\bigoplus^{r+1} \mathcal{O}_X \longrightarrow \mathcal{S}_1 \otimes \mathcal{L}'$$
This gives an $\underset{\mathcal{O}_X\text{-lin}}{\text{surjection}}$ $\underline{\mathrm{Sym}}^\bullet(\mathcal{O}_X^{r+1}) \longrightarrow \bigoplus \mathcal{S}_n \otimes \mathcal{L}'^n = \mathcal{S}'$

and Thus a closed embedding of sch$_X$.
$$\begin{tikzcd}
\underline{\mathrm{Proj}}(\mathcal{S}') \arrow[r, hook] \arrow[dr] & \underline{\mathrm{Proj}}(\underline{\mathrm{Sym}}^\bullet(\mathcal{O}_X^{r+1})) = X \times \mathbb{P}^r \arrow[d] \\
& X
\end{tikzcd}$$
But $\underline{\mathrm{Proj}}(\mathcal{S}') = \underline{\mathrm{Proj}}(\mathcal{S})$.

Def. Given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. The blow up of X along $\mathcal{I}$ is The X-scheme
$$\pi : \underline{\mathrm{Proj}}\Big(\bigoplus_{n \in \mathbb{N}} \mathcal{I}^n\Big) \longrightarrow X. \quad \text{set } \mathrm{Bl}_{\mathcal{I}}(X) = \underline{\mathrm{Proj}}\Big(\bigoplus_{n \in \mathbb{N}} \mathcal{I}^n\Big)$$
$$\bigoplus_{n\in\mathbb{N}} \mathcal{I}^n = (\mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots)$$

Notation: Given $f : Y_1 \to Y_2$ and an ideal sheaf $\mathcal{J}_2 \subseteq \mathcal{O}_{Y_2}$
$$f^{-1}(\mathcal{J}_2)\mathcal{O}_{Y_2} = \mathcal{J}_2 \mathcal{O}_{Y_2}$$
is The ideal sheaf obtained as The image of The map $f^*(\mathcal{J}_2) \to \mathcal{O}_{Y_2}$

Thm. (i) Given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, consider The blow-up map
$$\pi : \mathrm{Bl}_{\mathcal{I}}(X) \longrightarrow X$$
Then $\pi^{-1}(\mathcal{I})\, \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)} \xrightarrow{\sim} \mathcal{O}(1)$

In particular $\mathcal{I}\, \mathrm{Bl}_{\mathcal{I}}(X)$ is invertible.

(ii) $\pi^{-1}(X \setminus V(\mathcal{I})) \longrightarrow X \setminus V(\mathcal{I})$ is an isom

(iii) Let $f: Z \to X$ be a scheme map such that $\mathcal{I}\mathcal{O}_Z$ is invertible. Then f uniquely factors as

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[dr, "f"'] & & Bl_{\mathcal{I}}(X) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

<u>Pf</u> (i) $\pi^{-1}(\mathcal{I})\,\mathcal{O}_{Bl_{\mathcal{I}}(X)} = \mathcal{I}(\mathcal{O}_X \oplus \mathcal{I} \oplus \cdots)^{\sim}$

check on affine opens in X

$= [(\oplus \mathcal{I}^n)(1)]^{\sim}$

$= \mathcal{O}(1)$

(ii) $\mathcal{I}|_{X \setminus V(\mathcal{I})} = \mathcal{O}_X$. (iii) See prop 7.14.

<u>Thm</u> (Resolving indeterminacy via blow-up).
$\mathcal{L}$ be an invertible sheaf on X, $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$
Let $U = \bigcup_{i=0}^{n} D_{s_i}$. Then we have a map

$$\varphi: U \to \mathbb{P}^n_A$$

There exists an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ s.t we have a factorization

$$\begin{tikzcd} & Bl_{\mathcal{I}}(X) \arrow[dl] \arrow[dr, dashed, "\tilde{\varphi}"] & \\ X \supseteq U \arrow[rr, "\varphi"'] & & \mathbb{P}^n_A \end{tikzcd}$$

End of 29.11.24 lecture